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Abstract
We study a relativistic charged Dirac particle moving in a rotating magnetic field. By using a time-dependent unitary transformation, the Dirac equation with the time-dependent Hamiltonian can be reduced to a Dirac-like equation with a time-independent effective Hamiltonian. Eigenstates of the effective Hamiltonian correspond to cyclic solutions of the original Dirac equation. The nonadiabatic geometric phase of a cyclic solution can be expressed in terms of the expectation value of the component of the total angular momentum along the rotating axis, regardless of whether the solution is explictly available. For a slowly rotating magnetic field, the eigenvalue problem of the effective Hamiltonian is solved approximately and the geometric phases are calculated. The same problem for a charged or neutral Dirac particle with an anomalous magnetic moment is discussed briefly.

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In quantum mechanics, the Schrödinger equation, even with a time-independent Hamiltonian, can be solved analytically only in a few cases. With a time-dependent Hamiltonian, the problem is more difficult and fewer examples are well studied. One of the well studied examples is a nonrelativistic neutral particle with spin and magnetic moment in a rotating magnetic field [1–5]. This simple example has received much attention because of its relevance to the problem of geometric phases [6–8], and also because exact solutions are available. In a recent work [9] we have studied a nonrelativistic charged particle moving in a rotating magnetic field, with or without a central potential. The case with a central potential can describe the valence electron of an alkaline atom or that of a hydrogen atom under the influence of an external magnetic field. The problem can be treated analytically and some exact solutions are available. The purpose of this paper is to extend the previous work to a relativistic charged Dirac particle (without a central potential). The main interest of the problem is to see how the previous results are changed by the relativistic effect.

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Consider a charged particle with spin $\frac{1}{2}$, moving in a rotating magnetic field. The motion is described by the Dirac equation. This equation can be written in a form similar to the Schrödinger one. Thus the ideas of cyclic motion, nonadiabatic geometric phase etc for the Schrödinger equation [7] also apply here. The Hamiltonian for this system is of course time dependent. As in the nonrelativistic case, we use a time-dependent unitary transformation to reduce the Dirac equation to a Dirac-like one with a time-independent effective Hamiltonian. This is equivalent to transforming the equation into a rotating frame [10] where the magnetic field is static. Thus the effective Hamiltonian in that frame is time independent. However, as emphasized in [9], the transformation is merely a mathematical technique, and the effective equation in the rotating frame (which is not an inertial system) does not describe a real physical problem. The results derived from this equation are not observable in the rotating frame.

As in the nonrelativistic case, it can be shown that eigenstates of the effective Hamiltonian correspond to cyclic solutions of the original Dirac equation. The nonadiabatic geometric phase of a cyclic solution can be expressed in terms of the expectation value of the component of the total angular momentum along the rotating axis, regardless of whether the solution is explictly available. For a slowly rotating magnetic field, some of the terms in the effective Hamiltonian can be treated as small perturbations and the eigenvalue problem of the remaining terms can be solved exactly. In this approximation, the geometric phases of the cyclic solutions can be calculated explicitly. The above described procedure can also be applied to the Dirac–Pauli equation [11] for a charged or neutral particle with an anomalous magnetic moment, except that the effective Hamiltonian is too complicated and its eigenvalue problem has not been solved.

We begin with the Dirac equation

$$\left[i\gamma^\mu \left(\partial_\mu + \frac{iq}{\hbar c} A_\mu \right) - \frac{Mc}{\hbar} \right] \Psi = 0 \tag{1}$$

where M and q are respectively the mass and electric charge of the particle and A_μ is the vector potential describing the external electromagnetic fields. The latter is chosen as

$$A_0 = 0 \qquad \mathbf{A}(t) = \tfrac{1}{2} \mathbf{B}(t) \times \mathbf{r}. \tag{2}$$

This vector potential produces the magnetic field $\mathbf{B}(t)$, which is chosen to be one with a constant magnitude B and rotating about some fixed axis at a constant angle θ_B and with a constant frequency ω. The rotating axis is chosen as the z axis of the coordinate system. The magnetic field is therefore

$$\mathbf{B}(t) = B\mathbf{n}(t) \qquad \mathbf{n}(t) = (\sin\theta_B \cos\omega t, \sin\theta_B \sin\omega t, \cos\theta_B). \tag{3}$$

We take $B > 0$ without loss of generality. Note that $\mathbf{A}(t)$ also generates a time-varying electric field. Thus we are indeed dealing with a time-varying electromagnetic field. However, the electric field does not enter the Dirac equation directly if the particle has no anomalous magnetic moment. We write the Dirac equation in the following form:

$$i\hbar \partial_t \Psi = H(t)\Psi \tag{4a}$$

where

$$H(t) = c\boldsymbol{\alpha} \cdot \left[\mathbf{p} - \frac{q}{c} \mathbf{A}(t) \right] + Mc^2 \gamma^0 \tag{4b}$$

and $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$. To solve the equation we define the orbit angular momentum (in units of $\hbar$) $\mathbf{l} = \mathbf{r} \times \mathbf{p}/\hbar$, the spin angular momentum $\mathbf{s} = \boldsymbol{\Sigma}/2$ where $\Sigma^i = i\epsilon^{ijk}\gamma^j\gamma^k/2$ and the total angular momentum $\mathbf{j} = \mathbf{l} + \mathbf{s}$, and then make a time-dependent unitary transformation

$$\Psi(t) = W(t)\Phi(t) \tag{5}$$

where

$$W(t) = \exp(-i\omega t j_z) \tag{6}$$

and j_z is the z-component of the total angular momentum $\boldsymbol{j}$. This transformation is a generalization of that used in solving the Schrödinger equation for a neutral [2] or charged [9] particle with spin in the rotating magnetic field. It is not difficult to show that

$$W^\dagger(t)H(t)W(t) = H(0). \tag{7}$$

Thus we obtain the following equation for Φ:

$$i\hbar\partial_t\Phi = H_{\text{eff}}\Phi \tag{8}$$

where the effective Hamiltonian

$$H_{\text{eff}} = H(0) - \hbar\omega j_z. \tag{9}$$

Since H_{eff} is time independent, equation (8) has the formal solution

$$\Phi(t) = U_{\text{eff}}(t)\Phi(0) \qquad U_{\text{eff}}(t) = \exp(-iH_{\text{eff}}t/\hbar). \tag{10}$$

With the obvious relation $\Psi(0) = \Phi(0)$, the time-dependent Dirac equation (4) is formally solved as

$$\Psi(t) = U(t)\Psi(0) \qquad U(t) = W(t)U_{\text{eff}}(t). \tag{11}$$

Since $U(t)$ involves no chronological product, this solution is convenient for practical calculations.

Now we show that eigenstates of the effective Hamiltonian correspond to cyclic solutions of equation (4). We take the initial condition

$$\Psi_i(0) = \varphi_i \tag{12}$$

where φ_i is an eigenstate of the effective Hamiltonian with eigenvalue E_i, and calculate $\Psi_i(T)$, where $T = 2\pi/\omega$ is the period of the rotating magnetic field. Here for convenience we use one subscript i to represent all the quantum numbers that is needed to specify an eigenstate. Obviously, $U_{\text{eff}}(t)\varphi_i = \exp(-iE_it/\hbar)\varphi_i$, valid for all t, and $W(T)\varphi_i = \exp(-i2\pi j_z)\varphi_i$. Because we can always expand φ_i as a linear combination of the eigenstates of j_z, we obtain

$$\Psi_i(T) = \exp(-iE_iT/\hbar - i\pi)\Psi_i(0). \tag{13}$$

Hence it is indeed a cyclic solution, and the total phase change in a period is

$$\delta_i = -E_iT/\hbar - \pi \qquad \text{mod } 2\pi. \tag{14}$$

To determine the dynamic phase, we should calculate

$$\langle H(t)\rangle_i \equiv (\Psi_i(t), H(t)\Psi_i(t)) = (\Psi_i(0), W^\dagger H(t)W\Psi_i(0)) = (\varphi_i, H(0)\varphi_i).$$

Because $H(0) = H_{\text{eff}} + \hbar\omega j_z$, we have

$$\langle H(t)\rangle_i = E_i + \hbar\omega\langle j_z\rangle_i. \tag{15}$$

Here $\langle j_z\rangle_i = (\varphi_i, j_z\varphi_i) = (\Psi_i(t), j_z\Psi_i(t))$ is the expectation value of j_z in the state $\Psi_i(t)$, and it is time independent. Note that $\langle H(t)\rangle_i$ is also independent of t. Thus the state $\Psi_i(t)$ is somewhat similar to a stationary state in a system with a time-independent Hamiltonian. The dynamic phase is

$$\beta_i = -\hbar^{-1}\int_0^T dt\,\langle H(t)\rangle_i = -E_iT/\hbar - 2\pi\langle j_z\rangle_i. \tag{16}$$

Therefore the nonadiabatic geometric phase is

$$\gamma_i = \delta_i - \beta_i = -\pi + 2\pi\langle j_z\rangle_i \qquad \text{mod } 2\pi \tag{17}$$

and is determined by $\langle j_z \rangle_i$. This is a relativistic generalization of the result for a nonrelativistic neutral [2] or charged particle [9], and has the same form as the corresponding nonrelativistic result. It is valid regardless of whether φ_i is explicitly available or not, and is convenient for approximate calculations if necessary.

Our next task is to find the eigenvalues and eigenstates of H_{eff}. Since the effective Hamiltonian is somewhat complicated, we have to make some approximation. We assume that ω is small such that the term $-\hbar\omega j_z$ in H_{eff} can be treated as a small perturbation. That is, we are considering a slowly rotating magnetic field. In the nonrelativistic case [9], the restriction is specifically $\omega \ll |q|B/2Mc$, and we have argued that this is in fact a rather loose restriction. From the following result for the energy levels we would see that the argument also holds in the relativistic case. We thus decompose H_{eff} as

$$H_{\text{eff}} = H_{\text{eff}}^0 + H_{\text{eff}}' \tag{18}$$

where

$$H_{\text{eff}}^0 = c\boldsymbol{\alpha} \cdot \left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}_0\right) + Mc^2\gamma^0 \tag{19}$$

whose eigenvalue problem will be solved exactly, and

$$H_{\text{eff}}' = -\hbar\omega j_z \tag{20}$$

which will be treated as a small perturbation. In equation (19) $\boldsymbol{A}_0 = \boldsymbol{A}(0)$. Note that $H_{\text{eff}}^0 = H(0)$. It is not difficult to show that

$$H_{\text{eff}}^0 = \exp(-\mathrm{i}\theta_B j_y) H_{\text{eff}}^z \exp(\mathrm{i}\theta_B j_y) \tag{21}$$

where

$$H_{\text{eff}}^z = c\boldsymbol{\alpha} \cdot \left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}_z\right) + Mc^2\gamma^0, \tag{22}$$

$\boldsymbol{A}_z = \frac{1}{2}\boldsymbol{B}_z \times \boldsymbol{r}$ and $\boldsymbol{B}_z = B\boldsymbol{n}_z = B(0, 0, 1)$. This is the Hamiltonian of a relativistic charged particle in a static uniform magnetic field along the z axis.

The eigenvalue problem of H_{eff}^z is easy. We write down the eigenvalue equation

$$H_{\text{eff}}^z \zeta = E^0 \zeta \tag{23}$$

and denote $\zeta = (u, v)^\tau$ where u and v are two-component spinors and the superscript τ denotes matrix transposition. In terms of the two-component spinors the above equation takes the form

$$\begin{aligned} c\boldsymbol{\sigma} \cdot \left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}_z\right)u &= (E^0 + Mc^2)v \\ c\boldsymbol{\sigma} \cdot \left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}_z\right)v &= (E^0 - Mc^2)u. \end{aligned} \tag{24}$$

One can solve the first for v, and substitute it into the second. Then an equation in u alone is obtained. It can be solved in the cylindrical coordinates (ρ, ϕ, z). The energy eigenvalues are

$$E_i^0 = E_{n_z n_\rho m m_s \pm}^0 = \pm\{(Mc^2)^2 + (\hbar c k_z)^2 + |q|B\hbar c[2n_\rho + |m| + 1 - \epsilon(q)(m + 2m_s)]\}^{1/2} \tag{25}$$

where we use a single subscript i to represent all the quantum numbers and the sign of the energy; $k_z = 2\pi n_z/d$ where d is a length in the z direction for box normalization and $n_z = 0, \pm 1, \pm 2, \ldots$; $n_\rho = 0, 1, 2, \ldots$ is a radial quantum number; $m = 0, \pm 1, \pm 2, \ldots$ and $m_s = \pm 1/2$; $\epsilon(q)$ is a sign function of q. The corresponding eigenfunctions are given by

$$u_i(\rho, \phi, z, s_z) = N_i \mathrm{e}^{-\alpha^2\rho^2/2}(\alpha\rho)^{|m|} L_{n_\rho}^{|m|}(\alpha^2\rho^2) \frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}} \frac{\mathrm{e}^{\mathrm{i}k_z z}}{\sqrt{d}} \chi_{m_s}(s_z) \tag{26}$$

where $\alpha = \sqrt{|q|B/2\hbar c}$, the $L_{n_\rho}^{|m|}$ are Laguerre polynomials [12], $\chi_{m_s}(s_z)$ is the eigenstate of s_z with eigenvalue m_s and

$$v_i(\rho, \phi, z, s_z) = \frac{c}{E_i^0 + Mc^2} \boldsymbol{\sigma} \cdot \left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}_z\right) u_i(\rho, \phi, z, s_z). \tag{27}$$

We do not write down the specific form of v_i because it is lengthy and not necessary for the subsequent calculations. We just mention that it consists of two terms. One is proportional to u_i, and the other involves the factor $e^{i(m-1)\phi}\chi_{m_s+1}(s_z)$ or $e^{i(m+1)\phi}\chi_{m_s-1}(s_z)$, depending on whether $m_s = -1/2$ or $1/2$, respectively. It should be remarked that neither m nor m_s is a good quantum number. A good quantum number related to them is $m_j = m + m_s$, which is the eigenvalue of j_z, a conserved quantity. The reason why we need the quantum number m_s is that for a given m_j there are two kinds of solution, corresponding to the two values of m_s. In the above solutions the normalization constants are

$$N_i = \alpha \left(\frac{E_i^0 + Mc^2}{E_i^0} \right)^{1/2} \left[\frac{n_\rho!}{\Gamma(n_\rho + |m| + 1)} \right]^{1/2}. \tag{28}$$

Note that the solutions in the above forms are not appropriate when $E_i^0 = -Mc^2$, which may happen when $n_z = 0, n_\rho = 0, m = \epsilon(q)|m|$ and $m_s = \epsilon(q)|m_s|$. Indeed, one cannot eliminate v from equation (24) in this case. Rather, one should eliminate u and solve the resulting equation for v. The solution reads

$$v_{00mm_s-}(\rho, \phi, z, s_z) = N'_{00mm_s-} e^{-\alpha^2\rho^2/2}(\alpha\rho)^{|m|} \frac{e^{im\phi}}{\sqrt{2\pi}} \frac{1}{\sqrt{d}} \chi_{m_s}(s_z) \tag{29}$$

$$u_{00mm_s-}(\rho, \phi, z, s_z) = -\frac{1}{2Mc}\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}_z\right) v_{00mm_s-}(\rho, \phi, z, s_z). \tag{30}$$

We do not write down the specific form for u_{00mm_s-} as before. The normalization constant in the above solution is

$$N'_{00mm_s-} = \frac{\sqrt{2}\alpha}{\sqrt{\Gamma(|m| + 1)}}. \tag{31}$$

Thus the equation (23) is completely solved. The solutions of this equation can also be found in [13], but in different forms. The reason why the solution with a specific energy level can have different forms is that the energy levels are degenerate. It seems that the solutions in our form are more explicit and convenient.

Now that equation (23) is solved, the eigenvalue problem of H_{eff}^0 becomes trivial. The eigenfunctions are

$$\varphi_i^0 = \exp(-i\theta_B j_y)\zeta_i \tag{32}$$

where $\zeta_i = (u_i, v_i)^\tau$, and the corresponding energy eigenvalues are still given by equation (25). We will use these φ_i^0 as the approximate eigenfunctions of H_{eff}. Of course the explicit functional form of φ_i^0 is complicated, but this is not necessary in practical calculations. The lowest-order corrections to the energy eigenvalues are given by the expectation values of H'_{eff} in the approximate eigenstates. The corrected energy levels are

$$E_i = E_i^0 - m_j\hbar\omega\cos\theta_B. \tag{33}$$

Now there is no degeneracy in the quantum numbers.

Our final task is to work out the geometric phase in a period for the ith state, that is, a state with the initial condition (12). Since we have only an approximate result φ_i^0 for the eigenstate φ_i, we can calculate the nonadiabatic geometric phase only approximately. The result is

$$\gamma_i = -m_j\Omega_B \qquad \text{mod } 2\pi \tag{34}$$

where $\Omega_B = 2\pi(1 - \cos\theta_B)$ is the solid angle subtended by the trace of the rotating magnetic field. For $\omega \ll |q|B/2Mc$, this has the same value as the corresponding nonrelativistic result [9]. In the above approximation, it can be shown that

$$(\Psi_i(t), \boldsymbol{j}\Psi_i(t)) = m_j(\sin\theta_B\cos\omega t, \sin\theta_B\sin\omega t, \cos\theta_B). \tag{35}$$

Therefore the total angular momentum precesses synchronously with the magnetic field and approximately at the same angle θ_B with the rotating axis. Then Ω_B is also (approximately) the solid angle subtended by the trace of the total angular momentum. The geometric nature of the result (34) is thus obvious. In the nonrelativistic case, both the orbit and spin angular momentum precess synchronously with the magnetic field [9]. Here only the total angular momentum does. This is the main difference caused by the relativistic effect.

Now we turn to the problem of a charged or neutral particle with anomalous magnetic moment. The Dirac–Pauli equation is

$$\left[i\gamma^\mu \left(\partial_\mu + \frac{iq}{\hbar c} A_\mu \right) - \frac{Mc}{\hbar} - \frac{1}{2} \frac{\mu_a}{\hbar c} \sigma^{\mu\nu} F_{\mu\nu} \right] \Psi = 0 \tag{36}$$

where μ_a is the anomalous magnetic moment, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$. This equation differs from equation (1) by the last term in the square bracket. When $q = 0$ it describes a neutral particle, otherwise it describes a charged one. With the previous A_μ, it can be recast in the Hamiltonian form:

$$i\hbar \partial_t \Psi = H(t) \Psi \tag{37a}$$

where

$$H(t) = c\boldsymbol{\alpha} \cdot \left[\boldsymbol{p} - \frac{q}{c} \boldsymbol{A}(t) \right] + Mc^2 \gamma^0 - \mu_a \gamma^0 \boldsymbol{\Sigma} \cdot \boldsymbol{B}(t) + i\mu_a \boldsymbol{\gamma} \cdot \boldsymbol{E}(t). \tag{37b}$$

Note that the electric field $\boldsymbol{E} = -c^{-1} \partial_t \boldsymbol{A}$ now enters the equation directly. We make the time-dependent unitary transformation (5), (6). With some algebra it can be shown that

$$W^\dagger(t) H(t) W(t) = H(0) \tag{38}$$

still holds in the present case. Thus the equation for Φ has the same form as given by equations (8) and (9), and all subsequent discussions until equation (17) are still valid. Unfortunately, $H(0)$ is too complicated even for a neutral particle and we have not been able to obtain any eigenstate of it.

In conclusion we have considered a relativistic charged particle moving in a rotating magnetic field. The Hamiltonian for such a system is time dependent. By making use of a time-dependent unitary transformation, the Dirac equation can be reduced to a Dirac-like equation with an effective Hamiltonian which is time independent. In this way we obtain a formal solution to the original Dirac equation, which determines the time evolution of an arbitrary initial state. The time-evolution operator in this formal solution, unlike that for a general time-dependent Hamiltonian, involves no chronological product, and thus is convenient for practical calculations. Any solution with one of the eigenstates of the effective Hamiltonian as an initial state is a cyclic solution. The nonadiabatic geometric phase in a period for such a solution can be expressed in terms of the expectation value of the component of the total angular momentum along the rotating axis. This is an exact relation, which holds regardless of whether the solution is explicitly available, and is convenient for approximate calculations whenever necessary. For a slowly rotating magnetic field, the eigenvalue problem of the effective Hamiltonian is solved approximately, and the geometric phases are calculated. The difference between the relativistic results and the corresponding nonrelativistic ones is discussed. We also briefly discussed the same problem for a relativistic particle with an anomalous magnetic moment.

Acknowledgments

References

[1] Landau L D and Lifshitz E M 1977 *Quantum Mechanics* 3rd edn (Oxford: Pergamon)
[2] Wang S-J 1990 *Phys. Rev.* A **42** 5107
[3] Wagh A G and Rakhecha V C 1992 *Phys. Lett.* A **170** 71
[4] Ni G-J, Chen S-Q and Shen Y-L 1995 *Phys. Lett.* A **197** 100
[5] Ni G-J and Chen S-Q 2000 *Advanced Quantum Mechanics* (Shanghai: Fudan University Press) (in Chinese)
[6] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[7] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
[8] Li H-Z 1998 *Global Properties of Simple Physical Systems—Berry's Phase and Others* (Shanghai: Shanghai Scientific and Technical) (in Chinese)
[9] Lin Q-G 2001 *Phys. Rev.* A **63** 012108
[10] Wagh A G and Rakhecha V C 1993 *Phys. Rev.* A **48** R1729
[11] Pauli W 1941 *Rev. Mod. Phys.* **13** 203
[12] Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series, and Products* (New York: Academic)
[13] Bagrov V G and Gitman D M 1990 *Exact Solutions of Relativistic Wave Equations* (Dordrecht: Kluwer)